1. Tutorial: Transfer Matrix method for 1-d Ising system

Consider a one-dimensional system of N Spins $\sigma_i = \pm 1$ (spin up/down) on a lattice with periodic boundary conditions, i.e. $\sigma_{N+1} \equiv \sigma_1$. We only consider next-neighbor interactions and hence the Hamiltonian of the system is given by

$$H = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i,$$

where $h$ is an external field. By writing the spins as vectors, i.e. $1\hat{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $-1\hat{e} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we may introduce the transfer Matrix $T$, which we define here by

$$\sigma_i T \sigma_{i+1} = \exp\left[ K \sigma_i \sigma_{i+1} + \frac{1}{2} B (\sigma_i + \sigma_{i+1}) \right],$$

where $K = \beta J$ is the coupling between spins and $B = \beta h$ describes the effect of the external field.

a) Write down the partition sum of the system and rewrite it using the transfer matrix $T$. Use the transfer matrix notation to perform the sums over all $\sigma_i$ and hence calculate the partition sum in terms of $T$.

b) Determine the eigenvalues of $T$. In the limit of a large system $N \to \infty$, give the free energy $f$ and the magnetization $m = -\frac{\partial f}{\partial h}$ in terms of $K$ and $B$. Interpret this result!

2. Tutorial: Mean field theory

Consider again an Ising system of N spins $\sigma_i = \pm 1$ on a lattice with the Hamiltonian

$$H = -J \sum_{(ij)} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i,$$

where $\sum_{(ij)}$ denotes a sum over all distinct nearest neighbor pairs.

a) We write the spin variables as $\sigma_i = m + (\sigma_i - m) = m + \delta\sigma_i$ by using the fluctuations $\delta\sigma_i$ about the mean value $m$. Use this expression to simplify the Hamiltonian by neglecting terms of order $\delta\sigma_i \delta\sigma_j$, (we assume that we are sufficiently far from the critical temperature $T_c$). This allows us to rewrite the Hamiltonian in terms of an effective magnetic field. Give the expression for this field.

b) Calculate the free energy density (per site) $f(h,T) = F/N$ of the system and determine $m$ from the equation of state $m = -\frac{\partial f}{\partial h}$.

c) Calculate the Helmholtz free energy density $a(m,T)$ using a Legendre transformation. Minimize $a$ with respect to $m$ to determine the thermodynamically stable solution. A sketch may be helpful.

3. Tutorial: Correlation function for Ising spin system

**Deadline: July 9, 2012, 12 am.**
Here we generalize the previously discussed Ising model of $N$ spins $\sigma_i = \pm 1$ by introducing an arbitrary interaction $J(i,j)$ between spins at sites $i$ and $j$ and a position dependent external field $h_i$ at site $i$. However, we still assume that $J(i,j)$ is short-ranged. Throughout this exercise we apply the mean field approximation and hence it holds

$$\langle \sigma_i \rangle = \tanh \left[ \beta \left( h_i + \sum_j J(i,j) \langle \sigma_j \rangle \right) \right],$$

and the density matrix of the system is given by

$$\rho = \frac{1}{Z} \exp \left[ \beta \sum_i \sigma_i \left( h_i + \sum_j J(i,j) \langle \sigma_j \rangle \right) \right].$$

a) Consider the space-dependent susceptibility $\chi(i,j) = \frac{\partial \langle \sigma_i \sigma_j \rangle}{\partial h_i}$ and correlation function $G(i,j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$. Show that $G(i,j) = k_B T \chi(i,j)$.

b) Show that the susceptibility $\chi(i,j) = \frac{1}{\cosh^2(\beta q J m)} \left( \beta \delta_{ij} + \beta \sum_j J(i,j) \frac{\partial \langle \sigma_i \rangle}{\partial h_i} \right)$ and from now on assume that there is no external field $h_i = 0$. To solve the resulting equation for $\chi(i,j)$, perform a Fourier transformation, i.e. rewrite the equation using $\chi(k)$ and $J(k)$, with wave vector $k$. (Hint: Use the convolution theorem. The Result is $\chi(k) = \frac{1}{\cosh^2 (\beta q J m)} (\beta + \beta J(k) \chi(k))$.)

c) Give an explicit expression for $\chi(k)$ and simplify it using the mean field equation defining $m$ to eliminate hyperbolic functions and assuming that $m \ll 1$. Finally, transform your result to get the position-space susceptibility and the correlation function for large distances. (Hint: The result is $\chi(x_i - x_j) = \frac{a^2}{4\pi K|\rho_{x_i} - x_j|} \exp \left( \frac{-|x_i - x_j|}{a} \right)$.)

4. Exercise: Real-space renormalization group transformation

Figure 1: First two construction steps of the 2d Sierpinski gasket and the convention for spin labeling are shown.

Consider an Ising model on a Fractal: the two-dimensional Sierpinski gasket (see Fig. 1). The Hamiltonian reads

$$H[\sigma] = K \sum_{\langle i,j \rangle} \sigma_i \sigma_j + B \sum_i \sigma_i + K_3 M_3[\sigma],$$

where $K$ and $B$ correspond to the coupling constant and the magnetic field in the Ising model as discussed in the Tutorials, and the additional coupling $K_3$, where $M_3[\sigma] = \sum_{<i,j,k>} \sigma_i \sigma_j \sigma_k$, where $\sum_{<i,j,k>}$ denotes the sum over three spins being part of an elementary triangle, e.g. 1, 4, 5 and 4, 2, 6. The spin variables may take values $\sigma = \pm 1$. The Hamiltonian of the top unit cell for the situation as depicted on the right side of in Fig. 1 is

$$H[\sigma] = K [\sigma_1 \sigma_4 + \sigma_5] + \sigma_2 (\sigma_4 + \sigma_6) + (\sigma_3 + \sigma_4)(\sigma_5 + \sigma_6) + \sigma_5 \sigma_6] + \frac{B}{2} (\sigma_1 + \sigma_2 + \sigma_3 + B (\sigma_4 + \sigma_5 + \sigma_6)) + K_3 (\sigma_1 \sigma_5 \sigma_5 + \sigma_2 \sigma_4 \sigma_6 + \sigma_3 \sigma_5 \sigma_6).$$

(1)
Our goal is to renormalize the Hamiltonian such that the decimated spins $\mu$ and the spins of the unit cell $\sigma$ are related by a RG transformation $T[\mu|\sigma]$, such that

$$\exp\{H[\mu]\} = \sum_{\sigma} T[\mu|\sigma] \exp\{H[\sigma]\}.$$ 

In order to achieve this we have to relate the fields $(K, B, K_3)$ of the unit cell to the corresponding fields on the renormalized lattice $(K', B', K'_3)$. This one RG step is sufficient to solve the problem! In the following we will first determine these renormalization couplings and in the second part use thes results to calculate some critical properties and the RG flow.

a) Decimation i. The Hamiltonian, Eq. 1, is a function of all six spins in the Sierpinski gasket's unit cell. Carry out the decimation by calculating the partition sum for the possible spin configurations $<\sigma_1\sigma_2\sigma_3>$:

$$Z[\sigma_1, \sigma_2, \sigma_3] = \sum_{<\sigma_1\sigma_2\sigma_3>} \exp\{H[\sigma]\}.$$ 

Note, there are eight different spin configurations $\{+++, \ldots, −−−\}$. $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$.

Result:

$$Z[\sigma_1, \sigma_2, \sigma_3] = e^{\frac{B}{2}(\sigma_1 + \sigma_2 + \sigma_3)} 2(\cosh(2K\sigma_1 + B)e^{-K+K_3(\sigma_1−\sigma_2−\sigma_3)} + \cosh(2K\sigma_2 + B)e^{-K+K_3(−\sigma_1+\sigma_2−\sigma_3)} + \cosh(2K\sigma_3 + B)e^{-K+K_3(−\sigma_1−\sigma_2+\sigma_3)} + \cosh(2K\sigma_1 + \sigma_2 + \sigma_3) + 3B)e^{3K+K_3(\sigma_1+\sigma_2+\sigma_3)}.$$ 

b) Decimation ii. Your result for $Z[\sigma_1, \sigma_2, \sigma_3]$ now allows you to explicitly write down four “conditional partition functions” $\delta_i$ for the remaining spin configurations $i = <\sigma_1\sigma_2\sigma_3 > \in \{+++, \ldots, −−−\}$. What are the symmetries between these four functions $\delta_i$ with respect to $K, B$ and $K_3$?

Results:

$$\delta_1 = e^{3B/2(3\cosh(2K + B)e^{-K+K_3} + \cosh(3K+3B)e^{3K+3K_3})}$$

$$\delta_2 = e^{B/2(3\cosh(2K + B)e^{-K+K_3} + \cosh(-2K + 4B)e^{-K+3K_3} + \cosh(2K + 3B)e^{3K+K_3})}$$

c) RG transformation coupling. The functions $\delta_i$ for the renormalized Hamiltonian are:

$$\{+++\} \quad \delta_1 = e^{3K' + \frac{1}{2}K'_3 + C'}$$

$$\{++−\} \quad \delta_2 = e^{−K' + \frac{1}{2}K'_3 + C'}$$

$$\{+−−\} \quad \delta_3 = e^{−K' − \frac{1}{2}K'_3 + C'}$$

$$\{−−−\} \quad \delta_4 = e^{3K' − \frac{1}{2}K'_3 + C'}.$$ 

Note that $C'$ is an additional constant that arises which has to be considered, but is not important in the following. Construct the renormalization couplings which relate the primed and the unprimed fields! As an example, convince yourself that with these definitions $e^{8K'} = \delta_1\delta_4/\delta_2\delta_3$ is a function independent of $K', K_3'$ and $C'$. Then use the conditional partition functions obtained in part b) and write down $\delta_1\delta_4/\delta_2\delta_3$ to obtain a recurrence relation of the RG transformation: it relates the unit cell fields $K, B, K_3$ to the renormalized field $K'$. Construct the remaining couplings for $B'$ and $K'_3$.

Hint: Find combinations of $\delta_i$ that represent $e^{4B'}$ and $e^{8K'_3}$ and then use the $\delta_i$s from b) again.

5. Exercise: Correlations and renormalization flow

We continue to consider the Sierpinski gasket using the definitions and results of the previous exercise.
a) Correlation length. Calculate the recursion relation of $K$ and $K'$ for the case $B = K_3 = 0$ and find it’s fixed points. We want to investigate the critical behavior of the system at low temperatures (the large $K$ limit). To obtain a solvable recursion relation in this limit, define $t = e^{-4K}$ and Taylor expand the recursion for small $t$ (i.e. large $K$) up to second order in $t$. Then start to iterate and find a formula for the $\ell$th iteration $t(\ell)$ up to second order in $t$. 

Results: $e^{4K'} = e^{4K}(1 - e^{-4K} + 4e^{-8K})(1 + 3e^{-4K})^{-1}$ and $t' = t + 4t^2 + O(t^3)$ and $t(\ell) = t + 4\ell t^2 + O(t^3)$. Under RG transformations the correlation length scales as $\xi(\ell) = \xi_0/2^\ell$, where the factor 2 is the length scaling factor. Assume that $\xi(\ell)$ is a finite number of order one for all $\ell \gg 1$. Applying this assumption calculate the correlation length $\xi_0$. From your result for $t(\ell)$ above, justify that $\ell \approx 1/(4e^{-4K})$. 

b) RG flow. Investigate the RG flow in the ($B,K_3$) plane for $K \to \infty$ and analyse the ensuing phase diagram. HowTo: Use the linearized recursion relations for $K,B,K_3$. Determine the Eigenvalues of this set of equations, they determine the parameter flows in the critical regions. The parameters with Eigenvalues $\lambda_M = 1$ are called marginal parameters – such may or may not be relevant in higher order analysis. An Eigenvalue $\lambda_R > 1$ indicates a “relevant” variable and $\lambda < 1$ an “irrelevant” variable. Determine the left Eigenvectors associated to the parameters $B$ and $K_3$ and, from these, obtain the scaling fields for the marginal and the relevant parameter, $q_M$ and $q_R$. This allows you finally to draw the phase diagram in the ($K_3,B$) plane: draw the lines $q_M,q_R$. Analyse the parameter flows in this (nonorthogonal) coordinate system ($q_M,q_R$) for the marginal line of fixed points ($K^* = \infty, q_R^* = 0, q_M^*$). Infer the polarization of the lattice.

c) Absence of a phase transition. (i) Show that there is no phase transition! To this end reconsider the conditional partition functions $\delta_i$. You may take $B = K_3 = 0$ again, but in contrast to the previous exercise, $K$ has to stay finite now. Show that the recurrence relation for $K$ is always decreasing. (ii) Interpret the $\ell$-level partition function:

$Z = \left( \cdots \left( \left( Z_3^0(K)Z_0(K') \right)^3 Z_0(K''') \right)^3 \cdots \right) Z_0(K^{(\ell)})$,

where $Z_i^0 = \delta_i \delta_3^i$ is related to the constant $C'$, as introduced in the previous exercise. Use a numerical tool like Mathematica (available on CIP-Pool computers) to plot the specific heat of the system, $c = T\partial^2(T \ln Z)/\partial T^2$. Start with the 1-level structure and then extend your program to include larger structures as far as you like.